Count Theory Theorem 1

The Decoupling of Count from Linear Extent

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Theorem 1 (Sublinear Growth of Extent)

Let N unit spheres be arranged in Face-Centered Cubic (FCC) packing, the densest possible arrangement for equal spheres in unbounded space. Let R denote the radius of the minimal spherical envelope containing the aggregate, measured from the centre of the first sphere.

Then:

(i) The radius R is related to the count N by:

$$R = \left(\frac{3}{4\pi\sqrt{2}}\right)^{1/3} N^{1/3}$$

(ii) The rate of increase of radius with respect to count is:

$$\frac{dR}{dN} = \frac{1}{3}\left(\frac{3}{4\pi\sqrt{2}}\right)^{1/3} N^{-2/3}$$

(iii) This rate vanishes in the limit:

$$\lim_{N\to\infty} \frac{dR}{dN} = 0$$

Corollary 1.1 (Decoupling of Count and Linear Magnitude)

Cardinality (N) and linear extent (R) are not proportionally coupled. While count increases without bound, the incremental contribution of each additional sphere to the linear extent of the aggregate diminishes asymptotically to zero.

Proof

Preliminary: FCC Packing Fraction

The Face-Centered Cubic lattice achieves the maximum packing fraction for equal spheres in three dimensions (Kepler conjecture, proved by Hales 2005):

$$\eta = \frac{\pi}{3\sqrt{2}} \approx 0.7405$$

This value represents the ratio of volume occupied by spheres to total volume of any region of the lattice, in the limit of large regions.

Part (i): Derivation of R(N)

Consider N spheres of unit diameter arranged in FCC packing.

The total volume occupied by the spheres is:

$$V_{\text{spheres}} = N \cdot \frac{4}{3}\pi\left(\frac{1}{2}\right)^3 = \frac{\pi N}{6}$$

By definition of the packing fraction, this volume relates to the envelope volume:

$$V_{\text{spheres}} = \eta \cdot V_{\text{envelope}}$$

Therefore the volume of the spherical envelope is:

$$V_{\text{envelope}} = \frac{V_{\text{spheres}}}{\eta} = \frac{\pi N}{6} \cdot \frac{3\sqrt{2}}{\pi} = \frac{N}{\sqrt{2}}$$

The envelope is a sphere of radius R, so:

$$V_{\text{envelope}} = \frac{4}{3}\pi R^3 = \frac{N}{\sqrt{2}}$$

Solving for R:

$$R^3 = \frac{3N}{4\pi\sqrt{2}}$$

$$R = \left(\frac{3}{4\pi\sqrt{2}}\right)^{1/3} N^{1/3}$$

Let $c = \left(\frac{3}{4\pi\sqrt{2}}\right)^{1/3} \approx 0.5527$

Then: $R = c \cdot N^{1/3}$ ∎

Part (ii): Rate of Change

Differentiating R with respect to N:

$$\frac{dR}{dN} = c \cdot \frac{1}{3}N^{-2/3} = \frac{c}{3}N^{-2/3}$$

Substituting the value of c:

$$\frac{dR}{dN} = \frac{1}{3}\left(\frac{3}{4\pi\sqrt{2}}\right)^{1/3} N^{-2/3} \approx 0.1842 \cdot N^{-2/3}$$

∎

Part (iii): Limit

Since $\frac{dR}{dN} = \frac{c}{3}N^{-2/3}$ and the exponent $-2/3 < 0$:

$$\lim_{N\to\infty} \frac{dR}{dN} = \lim_{N\to\infty} \frac{c}{3}N^{-2/3} = 0$$

∎

Corollary 1.1: Interpretation

The theorem establishes that in geometric (non-linear) counting:

1. **Count grows linearly**: N = 1, 2, 3, 4, ...
2. **Extent grows sublinearly**: R ∝ N^(1/3)
3. **Rate of extent increase vanishes**: dR/dN → 0

This constitutes a formal decoupling: the cardinality of a collection and its linear spatial extent are governed by different growth laws. The conventional identification of "more" with "proportionally larger" is an artifact of linear (one-dimensional) counting, not an intrinsic property of enumeration itself.

∎

Remarks

Remark 1. The constant $c = \left(\frac{3}{4\pi\sqrt{2}}\right)^{1/3}$ arises purely from the geometry of optimal sphere packing. It is not an empirical fit but a derivation from first principles.

Remark 2. The choice of FCC packing is not arbitrary. It represents the densest possible arrangement (η ≈ 0.7405) and is crystallographically standard. The theorem thus applies to the most efficient geometric counting possible in three dimensions.

Remark 3. The result generalises to d dimensions. For optimal sphere packing in d dimensions with packing fraction η_d, the envelope radius satisfies R ∝ N^(1/d), and thus dR/dN ∝ N^(-(d-1)/d) → 0.

Remark 4. The vanishing of dR/dN as N → ∞ implies a kind of "finitude" in geometric counting: while count is unbounded, its expression as linear extent becomes asymptotically negligible. This inverts the conventional intuition that counting implies infinite extensibility.

Summary

Quantity	Symbol	Relationship	Behaviour as N → ∞
Count	N	Independent variable	→ ∞
Envelope radius	R	c·N^(1/3)	→ ∞ (sublinearly)
Rate of radius increase	dR/dN	(c/3)·N^(-2/3)	→ 0
Envelope volume	V	N/√2	→ ∞ (linearly with N)

Where $c = \left(\frac{3}{4\pi\sqrt{2}}\right)^{1/3} \approx 0.5527$

_This theorem forms the mathematical foundation of Count Theory, demonstrating that cardinality and linear magnitude are separable concepts._Start writing here...